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Mathematical Basis for the Plasma Kinetic Equations (BBGKY)

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The general family of kinetic equations, which in plasma kinetics are called the BBGKY equations, are obtained rigorously from basic probabilistic considerations in order to exhibit explicitly the conditions or assumptions under which they obtain.

Key Words: BBGKY hierarchy equations, probabilistic basis for kinetic equations.

1. Introduction

The plasma kinetic equations, referred to as the BBGKY hierarchy, are fundamental in the analysis of plasma kinetics. In the following, these equations are derived rigorously in order to determine explicitly the conditions or assumptions under which they are valid. The heuristic derivations do not clearly indicate the assumptions which are important to both theoretical and experimental plasma research. Since the functions in the BBGKY equations are probability density functions the following treatment is probabilistic.

2. Preliminaries

Consider first Euclidean 6-space E^6 in which a point is represented by the 6-tuple of real numbers (or coordinates) $\{q_1, q_2, q_3, p_1, p_2, p_3\}$ and n points are represented by n 6-tuples $\{q_1^s, q_2^s, q_3^s, p_1^s, p_2^s, p_3^s\}$ for $s=1 \ldots n$ or more concisely for each $s=1 \ldots n$ by $\{q_i^s\, p_i^s\}$ where $i=1, 2, 3$. Equivalently these n points may be represented in Euclidean $6n$-space E^{6n} by one point, or $6n$-tuple, $\{q_i^s, p_i^s\}$ where $s=1 \ldots n$, $i=1, 2, 3$.

Consider also a set of $6n$ elements $Q_1 \ldots Q_{3n}P_1 \ldots P_{3n}$ called observables; the entire set $Q_1 \ldots Q_{3n}P_1 \ldots P_{3n}$ being called the System S of observables. We then make the definition:
DEFINITION: *A (static) state* P *of the System S is an assignment* $\mathrm{P}: \mathrm{Q}_1 \ldots \mathrm{Q}_{3n}\mathrm{P}_1 \ldots \mathrm{P}_{3n} \rightarrow \{\mathrm{q}_1^1 \ldots \mathrm{q}_3^n, \mathrm{p}_1^1 \ldots \mathrm{p}_3^n\}$ *i.e., an assignment of* 6n *numerical values to the observables or an assignment of a point in* E^{6n}. *The correspondence between the* P's *and the points* $\{\mathrm{q}_i^s, \mathrm{p}_i^s\}$ *is assumed to be 1:1 so that we may write* $\mathrm{P}=\{\mathrm{q}_i^s, \mathrm{p}_i^s\}$.

The statistics of the System (in this static case) are given by a probability distribution F on the Borel Field of E^{6n} such that the following assumption holds:
ASSUMPTION 1: F *is absolutely continuous with respect to Lebesgue measure and thus there exists a measurable function* $\rho=\rho(\mathrm{q}_i^s, \mathrm{p}_i^s)$; *i.e., a probability density function, such that for any set* B *in the Borel Field one has:*

$$\mathrm{F(B)}=\int_{\mathrm{B}} \rho(\mathrm{q}_i^s, \mathrm{p}_i^s)(\mathrm{dq\,dp})_n \qquad \textit{where } (\mathrm{dq\,dp})_n=\mathrm{dq}_1^1 \ldots \mathrm{dq}_3^n\, \mathrm{dp}_1^1 \ldots \mathrm{dp}_3^n.$$

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In order to introduce dynamics into this schema, we specify the state concept in the dynamic case as follows:

DEFINITION: *A state function* P *of the System* S *is a mapping from the direct product* $S \times R^{\#}$, *where* $R^{\#}$ *is the real numbers, into* E^{6n} *such that for each* $t \epsilon R^{\#}$, P *provides an assignment* P(t) *of a point in* E^{6n} *to the observables in* S, *that is, for each* t, P *specifies a (static) state* $P(t) : Q_1 \ldots Q_{3n} P_1 \ldots P_{3n} \rightarrow \{q_i^s(t), p_i^s(t)\}$ *which by the preceding convention we may write* $P(t) = \{q_i^s(t), p_i^s(t)\}$.

We shall also make the following assumptions:

ASSUMPTION 2: *For given* t *and arbitrary* $\{\bar{q}_i^s, \bar{p}_i^s\} \epsilon E^{6n}$ *there exists a state function* P *such that* $P(t) = \{q_i^s(t), p_i^s(t)\} = \{\bar{q}_i^s, \bar{p}_i^s\}$.

ASSUMPTION 3: *The dynamics of the System* S *are given by a Hamiltonian function* $H_n = H_n(\{q_i^s, p_i^s\})$ *and by a one-parameter group* $\{U_t\}$, $t \epsilon R^{\#}$, *of contact transformations, differentiable with respect to* t, *so that if* P *is an arbitrary given state function then for* $t \epsilon R^{\#}$, $\tau \epsilon R^{\#}$

$$U_\tau P(t) = U_\tau \{q_i^s(t), p_i^s(t)\} = \{q_i^s(t+\tau), p_i^s(t+\tau)\} = P(t+\tau).$$

The statistics of the dynamical System are now given, for each value of the parameter $t \epsilon R^{\#}$, by a probability distribution F_t which for each $t \epsilon R^{\#}$ satisfies assumption 1 above so that we may write: $F_t(B) = \int_B \rho_t(q_i^s, p_i^s)(dq\,dp)_n$.

DEFINITION: *The System* S *in the state* P(t) *will be said to be in the Borel set* B *at* t *if* $P(t) = \{q_i^s(t), p_i^s(t)\} \epsilon B$. *In these terms* $F_t(B)$ *may be referred to as the probability that at* $t \epsilon R^{\#}$ *the System will be in a state in* B.

Since the variation of the states $P(t)$, or of the state function P, with t is given by assumption 3, it is possible to describe the related variation of the $F_t(B)$ with t as follows. By assumption 3: $U_\tau P(t) = P(t+\tau)$, the states $P(t)$ which are in a Borel set B at t will become states $P(t+\tau)$ in a Borel set B^τ at $t+\tau$. The set B^τ is the set of all points $\{\bar{q}_i^s, \bar{p}_i^s\}$ of E^{6n} which satisfy $\{\bar{q}_i^s, \bar{p}_i^s\} = \{q_i^s(t+\tau), p_i^s(t+\tau)\} = U_\tau\{q_i^s(t), p_i^s(t)\}$ for $\{q_i^s(t), p_i^s(t)\} = \{q_i^s, p_i^s\} \epsilon B$. Since the U_t are contact transformations and since the Poincare Invariants under contact transformations include the Lebesgue measure $\mu(B)$ of the Borel sets $B \subset E^{6n}$, we have $\mu(B^\tau) = \mu(B)$.

For given t and Borel set B with state $\{q_i^s(t), p_i^s(t)\} = \{q_i^s, p_i^s\} \epsilon B$ consider $\rho_t(q_i^s, p_i^s)\mu(B) = \rho(q_i^s(t), p_i^s(t), t)\mu(B)$ and similarly for τ and B^τ with $\{q_i^s(t+\tau), p_i^s(t+\tau)\} = \{\bar{q}_i^s, \bar{p}_i^s\} \epsilon B^\tau$ consider $\rho_{t+\tau}(\bar{q}_i^s, \bar{p}_i^s)\mu(B^\tau) = \rho(q_i^s(t+\tau), p_i^s(t+\tau))\mu(B^\tau)$.

ASSUMPTION 4: *The functions* $\rho_t(q_i^s, p_i^s) = \rho(q_i^s(t), p_i^s(t), t)$ *are differentiable with respect to* q_i^s, p_i^s *and* t *and* $\frac{d\rho}{dt}$ *(which exists by assumed differentiability of the contact transformations* U_t*) is uniformly continuous over each Borel set* B.

Then for $\tau = \Delta t$, sufficiently small, and arbitrary closed bounded Borel set $\beta \subset B$ one can write by the law of the Mean:

$$\begin{aligned}\rho(q_i^s(t+\Delta t), p_i^s(t+\Delta t), t+\Delta t)\mu(\beta^{\Delta t}) &= \rho(q_i^s(t+\Delta t), p_i^s(t+\Delta t), t+\Delta t)\mu(\beta) \\ &= \rho(q_i^s(t), p_i^s(t), t)\mu(\beta) + \Delta t \frac{d\rho}{dt}(q_i^s(t+\theta_1\Delta t), p_i^s(t+\theta_2\Delta t), t+\theta_3\Delta t)\mu(\beta)\end{aligned}$$

and using uniform continuity of $\frac{d\rho}{dt}$ one can write

$$\lim_{\Delta t \to 0} \frac{F_{t+\Delta t}(B^{\Delta t}) - F_t(B)}{\Delta t} = \int_B \frac{d\rho}{dt}(q_i^s, p_i^s, t)(dq\,dp)_n.$$

Letting $\frac{dF_t(B)}{dt}$ represent the left hand expression we make

ASSUMPTION 5: $\frac{dF_t(B)}{dt}=0$ *for all Borel sets* B. *From this assumption and continuity of* $\frac{d\rho}{dt}$ *we have*

$$0=\frac{\partial\rho}{\partial t}+\sum_{i=1}^{3}\sum_{s=1}^{n}\left(\frac{\partial\rho}{\partial q_i^s}\left(\frac{dq_i^s}{dt}\right)+\frac{\partial\rho}{\partial p_i^s}\left(\frac{dp_i^s}{dt}\right)\right)$$

$$=\frac{\partial\rho}{\partial t}+\sum_{i,s}\left(\frac{\partial\rho}{\partial q_i^s}\left(\frac{\partial H_n}{\partial p_i^s}\right)-\frac{\partial\rho}{\partial p_i^s}\left(\frac{\partial H_n}{\partial q_i^s}\right)\right)$$

$$=\frac{\partial\rho}{\partial t}+[\rho, H_n]. \qquad \text{[Liouville's Theorem]}$$

3. Derivation of BBGKY Equations

Consider a subspace $E^\sigma\subset E^{6n}$ where E^σ is the subspace of 6σ-tuples $\{q_i^\omega, p_i^\omega\}$, $\omega=j_1, j_2 \ldots j_\sigma$, the j_k being σ distinct integers from 1 to n; for purposes of simplicity in notation take $\omega=1,2,\ldots\sigma$.

For B_σ a Borel subset of E^σ let C_σ be the $6n$ dimensional "cylinder" on B_σ i.e., $C_\sigma=\{\{q_i^s, p_i^s\}$, $i=1, 2, 3$, $s=1 \ldots n$; such that for $s=1 \ldots \sigma$ $\{q_i^s, p_i^s\}\subset B_\sigma\}$. We shall write $C_\sigma=B_\sigma\times E'_\sigma$ where $E'_\sigma=\{\{q_i^s, p_i^s\}, s=\sigma+1, \ldots n\}$.

Since C_σ is a Borel subset of E^{6n} we have as above

$$F_t(C_\sigma)=\int_{C_\sigma}\rho_t(q_i^s, p_i^s)(dq\,dp)_n$$

$$=\int_{B_\sigma}\left[\int_{E'_\sigma}\rho_t(q_i^s, p_i^s)(dq_i^{\sigma+1}\ldots dp_i^n)\right](dq_i^1\ldots dp_i^\sigma).$$

Writing $$\rho_\sigma(q_i^s, p_i^s, t)=\int_{E'_\sigma}\rho_t(q_i^s, p_i^s)(dq_i^{\sigma+1}\ldots dp_i^n)$$

then $$F_t(C_\sigma)=\int_{B_\sigma}\rho_\sigma(q_i^s, p_i^s, t)(dq_i^1\ldots dp_i^\sigma).$$

The Hamiltonian $H_n=H_n(\{q_i^s, p_i^s\})$ will be of the form in the following assumption:

ASSUMPTION 6:. $$H_n=\left(\sum_{k=1}^{n}\sum_{i=1}^{3}\frac{(p_i^k)^2}{2m}\right)+V(\{q_i^s\})$$

where $$V(\{q_i^s\})=\sum_{k=1}^{n}\sum_{l>k}^{n}\theta_{k,l}+\sum_{k=1}^{n}\left[\Phi_w^k(\{q_i^s\})+\Psi_+^k(\{q_i^s\})\right]$$

with $\theta_{kl}=\theta_{lk}=\theta(q_i^k, q_i^l)=$ *a symmetric function of* q_i^k, q_i^p; *for example:* $\theta_{kl}=e^2\left[\sum_{i=1}^{3}(q_i^k-q_i^l)^2\right]^{-1/2}$ *and* $\Phi_w^k(\{q_i^s\})=$ *potential at* kth *point (or particle) due to the boundaries (in* E^6*) which are assumed to contain the* n *points (or particles) and* $\Psi_+^k(\{q_i^s\})=$ *potential at the* kth *point (or particle, or electron) due to a fixed background electric field (produced by + charges).*

One can then write

$$\frac{dF_t(C_\sigma)}{dt}=\int_{B_\sigma}\int_{E'_\sigma}\frac{d\rho}{dt}(q_i^s(t), p_i^s(t), t)(dq\,dp)_n$$

$$=\int_{B_\sigma}\int_{E'_\sigma}\left[\frac{\partial\rho}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial\rho}{\partial q_i^k}\left(\frac{p_i^k}{m}\right)-\frac{\partial\rho}{\partial p_i^k}\left(\frac{\partial V}{\partial q_i^k}\right)\right)\right](dq\,dp)_n$$

$$=\int_{B_\sigma}\frac{\partial}{\partial t}\int_{E'_\sigma}\rho+\int_{B_\sigma}\sum_{k=1}^{\sigma}\left(\frac{p_i^k}{m}\right)\frac{\partial}{\partial q_i^k}\int_{E'_\sigma}\rho+\int_{B_\sigma}\int_{E'_\sigma}\sum_{k=\sigma+1}^{n}\frac{\partial\rho}{\partial q_i^k}\frac{p_i^k}{m}$$

$$-\int_{B_\sigma}\sum_{k=1}^{\sigma}\left(\sum_{l>k}^{\sigma}\frac{\partial\theta_{kl}}{\partial q_i^k}+\sum_{l<k}\frac{\partial\theta_{kl}}{\partial q_i^k}+\frac{\partial\Phi^k{}_w}{\partial q_i^k}+\frac{\partial\Psi^k{}_+}{\partial q_i^k}\right)\frac{\partial}{\partial p_i^k}\int_{E'_\sigma}\rho$$

$$-\int_{B_\sigma}\int_{E'_\sigma}\sum_{k=1}^{\sigma}\left(\frac{\partial\rho}{\partial p_i^k}\right)\sum_{l>\sigma}^{n}\left(\frac{\partial\theta_{kl}}{\partial q_i^k}\right)$$

$$-\int_{B_\sigma}\int_{E'_\sigma}\sum_{k=\sigma+1}^{n}\left(\frac{\partial\rho}{\partial p_i^k}\right)\left(\sum_{l>k}^{n}\frac{\partial\theta_{kl}}{\partial q_i^k}+\sum_{m<k}\frac{\partial\theta_{mk}}{\partial q_i^k}+\frac{\partial\Phi_w^k}{\partial q_i^k}+\frac{\partial\Psi_+^k}{\partial q_i^k}\right).$$

In the above and following we omit the (dq, dp) notations whenever they are obvious.

Since
$$\theta_{kl}=\theta_{lk}\text{ and }\sum_{l>k}^{\sigma}\frac{\partial\theta_{kl}}{\partial q_i^k}+\sum_{m<k}\frac{\partial\theta_{mk}}{\partial q_i^k}=\sum_{\substack{l=1\\l\neq k}}^{\sigma}\frac{\partial\theta_{kl}}{\partial q_i^k}$$

the above may be written, using the definition of ρ_σ:

$$=\int_{B_\sigma}\left[\frac{\partial\rho_\sigma}{\partial t}+\sum_{k=1}^{\sigma}\left(\frac{\partial\rho_\sigma}{\partial q_i^k}\right)\left(\frac{p_i^k}{m}\right)-\sum_{k=1}^{\sigma}\left(\frac{\partial\rho_\sigma}{\partial p_i^k}\right)\left(\sum_{\substack{l=1\\l\neq k}}^{\sigma}\frac{\partial\theta_{kl}}{\partial q_i^k}\right)\right]$$

$$-\int_{B_\sigma}\sum_{k=1}^{\sigma}\frac{\partial}{\partial p_i^k}\cdot\sum_{l>k}^{n}\int_{E'_\sigma}\left(\frac{\partial\theta_{kl}}{\partial q_i^k}\right)\rho-\int_{B_\sigma}\sum_{k=1}^{\sigma}\frac{\partial\rho_\sigma}{\partial p_i^k}\left(\frac{\partial\Phi_w^k}{\partial q_i^k}+\frac{\partial\Psi_+^k}{\partial q_i^k}\right)$$

$$+\int_{B_\sigma}\int_{E'_\sigma}\sum_{k=\sigma+1}^{n}\left(\frac{\partial\rho}{\partial q_i^k}\right)\left(\frac{p_i^k}{m}\right)$$

$$-\int_{B_\sigma}\int_{E'_\sigma}\sum_{k=\sigma+1}^{n}\left(\frac{\partial\rho}{\partial p_i^k}\right)\left(\sum_{\substack{l=1\\l\neq k}}^{n}\frac{\partial\theta_{kl}}{\partial q_i^k}+\frac{\partial\Phi_w^k}{\partial q_i^k}+\frac{\partial\Psi_+^k}{\partial q_i^k}\right)$$

and one can abbreviate the notation for these five integrals as:
$=I(1)-I(2)-I(3)+I(4)-I(5)$.

Now in the above notation for $E^\sigma=\{\{q_i^k, p_i^k\}; k=1\ \ldots\ \sigma\}$ letting $E^{(l)}=\{\{q_i^k, p_i^k\}; k=l\}$ we set $E^{\sigma,l}=E^\sigma\dot{+}E^{(l)}$, that is, the minimum subspace containing the subspaces E^σ, $E^{(l)}$. Also, as above, let $E'_\sigma=$ the complementary subspace of E^σ in E^{6n} consisting of all $6(n-\sigma)$-tuples $\{q_i^k, p_i^k\}$ where $k=\sigma+1, \sigma+2, \ldots\ n$ and $i=1, 2, 3$ and let $E'_{\sigma,l}=$ the complementary subspace of $E^{\sigma,l}$ in E^{6n} consisting of all $6(n-\sigma-1)$-tuples $\{q_i^k, p_i^k\}$ where $k\neq l$, $\sigma+1\leq k\leq n$ and $i=1, 2, 3$. Also similar to the definition of ρ_σ above we shall let:

$$\rho_{\sigma,l}(q_i^s, p_i^s, t)=\int_{E'_{\sigma,l}}\rho(q_i^s, p_i^s, t)(dq\,dp)'_{\sigma,l}$$

where $(dq\,dp)'_{\sigma,l}$ represents the usual "volume" element in $E'_{\sigma,l}$.

Furthermore we now make the following:

ASSUMPTION 7: $\rho(q_i^s, p_i^s, t)$ *is a function symmetric with respect to the indices* s.

It is clear then that the functions $\rho_{\sigma+1}(q_i^s, p_i^s, t) = \rho_{\sigma, \sigma+1}(q_i^s, p_i^s, t)$ and $\rho_{\sigma, 1}(q_i^s, p_i^s, t)$ differ only in that the coordinates $q_i^{\sigma+1}$, $p_i^{\sigma+1}$ occurring in $\rho_{\sigma+1}$ are replaced by q_i^l, p_i^l in $\rho_{\sigma, l}$. Thus

$$I(2) = \int_{B_\sigma} \sum_{k=1}^{\sigma} \frac{\partial}{\partial p_i^k} \sum_{l>k}^{n} \int_{E'_\sigma} \frac{\partial \theta_{kl}}{\partial q_i^k} \rho$$

$$= \int_{B_\sigma} \sum_{k=1}^{\sigma} \frac{\partial}{\partial p_i^k} \sum_{l>k}^{n} \int_{E^{(l)}} \frac{\partial \theta_{kl}}{\partial q_i^k} \int_{E'_{\sigma, \rho}} \rho(q_i^s, p_i^s, t)\, (dq\, dp)'_{\sigma, 1}\, (dq_i^l, dp_i^l)$$

$$= \int_{B_\sigma} \sum_{k=1}^{\sigma} \sum_{l>k}^{n} \int_{E^{(l)}} \frac{\partial \theta_{kl}}{\partial q_i^k} \frac{\partial}{\partial p_i^k} (\rho_{\sigma, 1}(q_i^s, p_i^s, t))\, dq_i^l\, dp_i^l$$

$$= \int_{B_\sigma} \sum_{k=1}^{\sigma} (n-\sigma) \int_{E^{(\sigma+1)}} \frac{\partial \theta_{k, \sigma+1}}{\partial q_i^k} \frac{\partial \rho_{\sigma+1}}{\partial p_i^k} (dq_i^{\sigma+1}\, dp_i^{\sigma+1})$$

since

$$\int_{E^{(\sigma+1)}} \frac{\partial \theta_{k, \sigma+1}}{\partial q_i^k} \frac{\partial}{\partial p_i^k} (\rho_{\sigma, \sigma+1}) = \int_{E^{(l)}} \frac{\partial \theta_{kl}}{\partial q_i^k} \frac{\partial}{\partial p_i^k} (\rho_{\sigma, l}).$$

Suppose now that the states of the System are restricted by:

ASSUMPTION 8: *There exists a closed bounded subset* Σ^{3n} *of* E^{3n} *such that the probability density function* $\rho(q_i^s, p_i^s, t)$ *is identically* 0 *for all* (q_i^s, p_i^s, t) *such that* $\{q_i^s\}$ $s=1 \ldots n$, $i=1, 2, 3$ *is a point on the boundary of* Σ^{3n} *or outside* Σ^{3n}. *This is equivalent to saying that for any* t *and any Borel set* B *with* $B \cap E^{3n}$ *in the closure of the complement of* Σ^{3n}, *the probability*, $F_t(B)$, *that the System is in a state in* B *is* 0.

We shall also take the boundary of Σ^{3n} to be the boundary pertaining to the potential functions Φ_w^k of assumption 6.

One can then write:

$$I(4) = \int_{B_\sigma} \int_{E'_\sigma} \sum_{k=\sigma+1}^{n} \left(\frac{\partial \rho}{\partial q_i^k}\right) \left(\frac{p_i^k}{m}\right) = \int_{B_\sigma} \sum_{k=\sigma+1}^{n} \int_{E'_{\sigma, k}} \left[\int_{-\infty}^{\infty} \left(\frac{p_i^k}{m}\right) \left[\int_{-\infty}^{\infty} \frac{\partial \rho}{\partial q_i^k}\, dq_i^k \right] dp_i^k \right] (dq\ dp)'_{\sigma, k}$$

and $\displaystyle\int_{-\infty}^{\infty} \frac{\partial \rho}{\partial q_i^k}\, dq_i^k = 0$ by virtue of assumption 8 hence $I(4) = 0$.

If also one makes:

ASSUMPTION 9: *For any given* $k=1 \ldots n$ $\rho(q_i^s, p_i^s, t) \to 0$ *as* $p_i^k \to \pm\infty$ then $I(5) = 0$ by the same type of argument.

Finally from assumption 5 $\dfrac{dF_+(C_\sigma)}{dt} = 0$ for any C_σ hence for any B_σ $0 = I(1) - I(2) - I(3)$ and since the integrands are assumed continuous we get the BBGKY hierarchy equations:

$$\frac{\partial \rho_\sigma}{\partial t} + \sum_{k=1}^{\sigma} \left(\frac{\partial \rho_\sigma}{\partial q_i^k}\right) \left(\frac{p_i^k}{m}\right) - \sum_{k=1}^{\sigma} \left(\frac{\partial \rho_\sigma}{\partial p_i^k}\right) \left(\sum_{\substack{l=1 \\ l \neq k}}^{\sigma} \frac{\partial \theta_{kl}}{\partial q_i^k} + \frac{\partial \Phi_w^k}{\partial q_i^k} + \frac{\partial \Psi_+^k}{\partial q_i^k} \right)$$

$$= \sum_{k=1}^{\sigma} (n-\sigma) \int_{E^{(\sigma+1)}} \frac{\partial \theta_{k, \sigma+1}}{\partial q_i^k} \frac{\partial \rho_{\sigma+1}}{\partial p_i^k} (dq_i^{\sigma+1}\ dp_i^{\sigma+1}).$$

If one takes the Hamiltonian $H_\sigma = H_\sigma(\{q_i^s, p_i^s\})s=1 \ldots \sigma$ to be

$$H_\sigma = \sum_{k=1}^{\sigma}\left(\sum_{i=1}^{3}\frac{(p_i^k)^2}{2m} + \sum_{l>k}^{\sigma}\theta_{kl} + \Phi_w^k(q_i^k) + \Psi_+^k(q_i^k)\right)$$

then these equations become

$$\frac{\partial\rho_\sigma}{\partial t} + [\rho_\sigma H_\sigma] = \sum_{k=1}^{\sigma}(n-\sigma)\int_{E^{(\sigma+1)}}\frac{\partial\theta_{k,\sigma+1}}{\partial q_i^k}\frac{\partial\rho_{\sigma+1}}{\partial p_i^k}(dq_i^{\sigma+1}dp_i^{\sigma+1}).$$

The conditions under which these BBGKY equations are valid are thus explicitly given by the assumptions 1 through 9.

4. More General Case

The preceding discussion is conditioned by its restriction to Systems of $6n$ observables Q_i, P_i $i=1 \ldots 3n$, that is, to exactly n points (particles) in E^6. Therefore the functions $F_T(B) = F(B, t)$ are, in this sense, conditional probability distributions. A somewhat more general situation may be treated as outlined briefly in the following.

Let E^ω be the linear space of all infinite double sequences:

$$\{q_1^1, q_2^1, q_3^1, q_1^2, q_2^2, \ldots p_1^1, p_2^1, p_3^1, p_1^2, p_2^2, \ldots\} = \{q_i^s, p_i^s\}\ i=1, 2, 3\ s=1, 2, \ldots.$$

If $E^6 = \{q_i, p_i\}$ $i=1, 2, 3$ then E^ω may also be regarded as the infinite direct product:

$$E^\omega = E^6 \times E^6 \times E^6 \times \ldots.$$

Similarly writing as above $E^\sigma = \{\{q_i^s, p_i^s\};\ i=1, 2, 3;\ s=1, 2 \ldots \sigma\}$ and B^σ for a Borel subset of E^σ then for $m \geqq \sigma$ we denote the cylinders C_σ^m in E^m and C_σ^ω in E^ω by: $C_\sigma^m = B_\sigma \times E_\sigma^{m'}$ where

$$E_\sigma^{m'} = \{\{q_i^s, p_i^s\};\ s=\sigma+1, \ldots m\} \text{ and } C_\sigma^\omega = B_\sigma \times E_\sigma^{\omega'} \text{ where } E_\sigma^{\omega'} = \{\{q_i^s, p_i^s\};\ s=\sigma+1, \ldots\}.$$

If the System may vary stochastically with respect to t we may let $P_m(t)$ be the probability that, for $t \epsilon R$, there are m points (particles) in E^6 or equivalently that the System consists of $6m$ observables Q_i, P_i. Also let $F_\sigma^m(C_\sigma^m, t)$ be the conditional probability that, if there are $6m$ observables in the System, then this System will be in a state $P = \{q_i^s, p_i^s\}$ such that the 6σ-tuple $\{q_i^s, p_i^s\}$ $s=1 \ldots \sigma$ will be in B_σ at t. Then the probability that the System will be in a state $P = \{q_i^s, p_i^s\}$ which is in B_σ at t is given by: $F_\sigma^\omega(C_\sigma^\omega, t) = \sum_{m \geqslant \sigma} P_m(t) F_\sigma^m(C_\sigma^m, t)$.

If for each m the distribution $F_\sigma^m(C_\sigma^m, t)$ is absolutely continuous with respect to Lebesgue measure in E^m then there exists as above a measurable function ρ_σ^m such that

$$F_\sigma^m(C_\sigma^m, t) = \int_{C_\sigma} \rho_\sigma^m(q_i^s, p_i^s, t)(dq\ dp)_m.$$

Furthermore $F_\sigma^\omega(C_\sigma^\omega, t)$ will also be absolutely continuous with respect to the product measure E^ω and $F_\sigma^\omega(C_\sigma, t) = \int_{C_\sigma} \rho_\sigma^\omega(q_i^s, p_i^s, t)(dq\ dp)_\omega$ where ρ_σ^ω is a function measurable with respect to the product measure in E^ω.

Since $F_\sigma^\omega(C_\sigma^\omega, t) = \sum_{m \geqslant \sigma} P_m(t) F_\sigma^m(C_\sigma^m, t)$ the sequence $\sum_{m \geqslant \sigma}^{n} P_m(t)\rho_\sigma^m(q_i^s, p_i^s, t)$, as $n \to \infty$, converges in measure to $\rho_\sigma^\omega(q_i^s, p_i^s, t)$ and under proper assumptions on the uniform continuity of the

derivatives of the terms of the sequence one may relate the hierarchy equations for the ρ_σ^ω to the above derived corresponding relations for the individual ρ_σ^m occurring in the terms of the sequence.

5. Summary

The preceding derivation of the BBGKY hierarchy equations and the specific assumptions on which it is based are not restricted to plasma kinetics. As indicated in the introduction the treatment is essentially probabilistic and nearly all the assumptions are probabilistic in nature. The excepted nonprobabilistic assumptions refer to the existence of a Hamiltonian and an associated contact transformation group, both of which are more general than the requirements of plasma kinetics.

6. References

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